J. Fluid Mech. (1970), *vol.* 42, *part* 2, *pp.* 257–267
Printed in Great Britain

The effects of a corner on a propagating internal gravity wave

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(Received 29 July 1969 and in revised form 4 December 1969)

A solution satisfying the usual radiation conditions is found to the problem of an internal wave propagating towards a corner. It is found that, far from the corner, and the characteristic emanating from the corner, the solution is asymptotically equivalent to the solution found by plane wave reflexions from an infinite wall. The present solution shows that, by imposing the radiation condition, a singularity predicted by the ray theory along the corner characteristic is absent. A further singularity in the present solution along the same characteristic is shown to be due to an inability of the usual linear internal wave equations to fully describe the motion. The solution is for restricted corner angles.

1. Introduction

In attempting to find the effect upon an internal wave propagating towards a sloping beach, it is of interest to examine the influence of the corner. Sandstrom (1966) has examined problems similar to this using purely ray methods, which led to anomalous energy flux radiations. In particular, when the solution obtained in this manner is expanded in terms of eigensolutions, it is found that there are modes present which are transporting energy in a direction opposite to that implied by the usual radiation condition. Wunsch (1969) has examined the eigensolutions, in a wedge, of the equations governing the propagation of internal waves. However, from this work, the effects of the corner on a propagating plane wave are not evident.

Here we examine this problem by supposing the corner is in an infinite medium. A certain form of the solution is assumed, so that only modes radiating energy in the appropriate direction are present. This results in an integral equation, which is solved and properties of the resulting flow fields are examined. Unfortunately, the solution is for restricted angles of the corner.

2. Equations of motion

The linear equations governing the two-dimensional propagation of small disturbances in a stable, density stratified, incompressible, inviscid fluid are

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p_1}{\partial x}, \tag{1}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p_1}{\partial y} - \rho_1 g, \tag{2}$$

$$\frac{\partial p_1}{\partial t} + v\frac{d\rho_0}{dy} = 0, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

where x and y are the horizontal and vertical co-ordinates. The quantities ρ_0, ρ_1, p_1, u, v are, respectively, the equilibrium density, perturbation density, perturbation pressure, horizontal and vertical velocities.

If the Boussinesq approximation is made, and the stream function $\hat{\psi}$ is introduced, such that

$$u = -\frac{\partial \hat{\psi}}{\partial y}, \quad v = \frac{\partial \hat{\psi}}{\partial x},$$

(1)–(4) imply

$$\nabla^2 \hat{\psi}_{tt} + N^2 \hat{\psi}_{xx} = 0, \tag{5}$$

where

$$N = \left(-\frac{g}{\rho_0}\frac{d\rho_0}{dy}\right)^{\frac{1}{2}}$$

is the Brunt–Väisälä frequency.

If the motion is periodic, so that

$$\hat{\psi} = \psi e^{-i\sigma t} \quad (\sigma > 0),$$

(5) becomes

$$\psi_{yy} - \gamma^2 \psi_{xx} = 0, \quad \gamma^2 = \frac{N^2}{\sigma^2} - 1. \tag{6}$$

Internal waves exist if $\sigma^2 < N^2$. We assume this to be the case, and also that N^2 is constant, i.e. the equilibrium density stratification is exponential. Under these conditions, plane wave solutions to (6) exist in the form,

$$\psi = \exp\left\{ik\left(\pm y \pm \frac{x}{\gamma}\right)\right\}, \quad (k \text{ real}). \tag{7}$$

3. The ray solution

It has been shown by Mowbray & Rarity (1967) that, for internal waves of the type given by (7), the group velocity is at right angles to the phase velocity, with the x component of both velocities in the same direction.

Consider a plane wave,

$$\exp\left\{ik\left(y - \frac{x}{\gamma}\right)\right\}, \quad (k > 0),$$

incident upon an infinite barrier situated at $x = \alpha y$, with $|\alpha| < \gamma$. This wave is introducing energy from $x = +\infty$, along the characteristics $y - (x/\gamma) =$ constant, and will be reflected. Using methods similar to geometrical optics (see Phillips 1966, p. 176), the reflected wave is

$$-\exp\left\{ik\left(\frac{\gamma - \alpha}{\gamma + \alpha}\right)\left(y + \frac{x}{\gamma}\right)\right\}.$$

Consider now the region bounded below by $y = 0$, and to the left by $y = x/\alpha$, with $|\alpha| < \gamma$ (see figure 1). The eigensolutions or modes satisfying (6) in a half plane with $\psi = 0$ on $y = 0$ are

$$\psi_{\pm k}(x,y) = \sin ky\, e^{\pm ik(x/\gamma)} \quad (k > 0). \tag{8}$$

For these solutions the sign in $e^{\pm ik(x/\gamma)}$ determines the direction of energy propagation. For $e^{+ik(x/\gamma)}$ the energy is propagating to the right, and for $e^{-ik(x/\gamma)}$ to the left. The problem to be posed is: given an incident wave,

$$\psi_I(x,y) = \sin \lambda y\, e^{-i\lambda(x/\gamma)} \quad (\lambda > 0),$$

introducing energy from $x = +\infty$ onto the barrier at $x = \alpha y$, what is the resulting reflected solution?

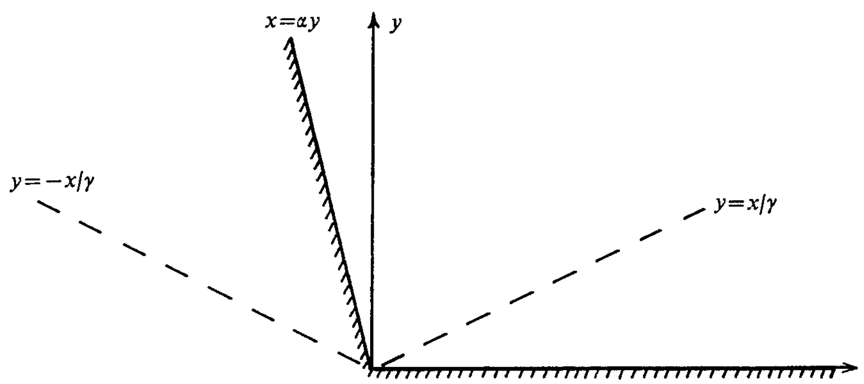

FIGURE 1. The geometry of the corner under consideration showing the characteristics originating at the corner.

A possible solution can be found by separating the incident wave into its two plane wave components, and reflecting these from the various walls using the method similar to geometrical optics discussed above. This shall be termed the ray solution, and is given by

$$\text{ray solution} = \begin{cases} \dfrac{1}{2i}\left[\exp\left\{ic^{-1}\lambda\left(-y+\dfrac{x}{\gamma}\right)\right\} - \exp\left\{ic\lambda\left(y+\dfrac{x}{\gamma}\right)\right\}\right], & (y > x/\gamma) \\ -\sin c\lambda y \exp\left(ic\lambda x/\gamma\right) & (y < x/\gamma), \end{cases}$$

where
$$c = \frac{\gamma-\alpha}{\gamma+\alpha}.$$

The ray solution may be expanded in terms of the eigensolutions (8) to yield

$$\text{ray solution} = \frac{1}{2\pi i}\int_\Gamma \sin \lambda\tau y \exp\left(i\lambda\tau\frac{x}{\gamma}\right)\left[\frac{1}{\tau-c^{-1}} - \frac{1}{\tau-c}\right] d\tau.$$

The integration path Γ is along the entire real axis indented above the point $\tau = c^{-1}$ and below the point $\tau = c$. That this integral is the ray solution is easily verified using contour integration. Thus, the ray solution consists of modes exporting energy to $x = +\infty$, corresponding above to $\tau > 0$, and modes introducing energy from $x = +\infty$, corresponding to $\tau < 0$. But the incident mode is introducing energy from $x = +\infty$, and we are seeking the reflected solution. On physical grounds the reflected solution cannot also be introducing energy from the far field. Hence, the ray solution does not satisfy the radiation condition, and cannot be accepted as the solution to the physical problem.

However, there seems no reason to reject our method of reflecting a plane wave from an infinite wall. The ray solution is found using this method, and therefore it would seem that deviations from this solution could only be caused by the presence of the corner. The effects of the corner should be felt near the corner and, because of the hyperbolic nature of the equations, near the characteristic emanating from the corner. Thus the assertion, which we shall show to hold, is that the ray solution is asymptotically valid at large distances from both the corner and the characteristic, $y = x/\gamma$, originating at the corner.

4. A solution satisfying the radiation condition

In §4 we look for a solution to the problem posed in §3 such that the radiation condition is satisfied, i.e. the reflected solution consists only of modes exporting energy to $x = +\infty$.

Given an incident mode,

$$\psi_I(x,y) = \sin \lambda y \, e^{-i\lambda(x/\gamma)} \quad \lambda > 0,$$

we can write the total stream function, ψ_T, as

$$\psi_T = \psi_I + \psi_R, \tag{9}$$

where ψ_R is the reflected solution. The reflected solution will automatically satisfy the radiation condition if it is represented by

$$\psi_R(x,y) = \int_0^\infty \frac{A(k)}{k} \sin ky \exp\left(ik\frac{x}{\gamma}\right) dk. \tag{10}$$

The boundary condition, imposed by the barrier situated at $x = \alpha y$ ($|\alpha| < \gamma$), is $\psi_T(\alpha y, y) = 0$ for all $y \geqslant 0$. Hence, from (9) we obtain the integral equation

$$0 = \sin \lambda y \exp\left(-\frac{i\lambda\alpha y}{\gamma}\right) + \int_0^\infty \frac{A(k)}{k} \sin ky \exp\left(\frac{ik\alpha y}{\gamma}\right) dk \tag{11}$$

for the unknown amplitude function $A(k)/k$.

Because the kernel of (11), i.e. $\sin ky \, e^{i\alpha ky/\gamma}$, is a function of the product ky it is convenient to take a Mellin transform with respect to y. This yields the transform $\bar{A}(s)$ of $A(k)$ which, upon simplification, is

$$\bar{A}(s) = -\frac{\sin[\frac{1}{2}\{s(\pi + iK)\}]\lambda^s}{\sin[\frac{1}{2}\{s(\pi - iK)\}]}, \tag{12}$$

where $$K = \ln\left(\frac{\gamma-\alpha}{\gamma+\alpha}\right) = \ln c.$$

In taking the transform of (11), the transforms of the known functions, viz. functions of the type e^{iay}, are valid in a restricted range of the transform variable.

For the variable in (12) this range of validity is $-1 < \mathrm{Re}\, s < 0$. When $|K|$ is large enough, $\bar{A}(s)$ has poles in this region of s. At present we shall assume that $\mathrm{Re}\, s$ lies to the right of these poles, so that the inversion path may be taken as the imaginary axis. In this case,

$$A(k) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \bar{A}(s)\, k^{-s} ds.$$

The integral is non-convergent at the end points, and has to be treated in a generalized sense. This is most easily achieved by transforming the path of integration to the real axis and then subtracting the singularity at infinity. The resulting integrals yield (see appendix):

$$\begin{aligned} A(k) &= \frac{\nu}{2\pi i}\left[\coth \nu\left(K+\ln\frac{k}{\lambda}\right)+\coth \nu\left((K-\ln\frac{k}{\lambda}\right)\right] \\ &\quad -\tfrac{1}{2}\left[\delta\left(K+\ln\frac{k}{\lambda}\right)+\delta\left(K-\ln\frac{k}{\lambda}\right)\right] \\ &= -\frac{\nu}{\pi i}\left[\frac{(k/\lambda)^{2\nu}(c^{2\nu}-c^{-2\nu})}{[(k/\lambda)^{2\nu}-c^{-2\nu}]\,[(k/\lambda)^{2\nu}-c^{2\nu}]}\right] \\ &\quad -\tfrac{1}{2}[\delta(K+\ln k/\lambda)+\delta(K-\ln k/\lambda)], \end{aligned}$$

where $\qquad \nu = \pi/(\pi - iK)$. Note $0 < \mathrm{Re}\,\nu \leqslant 1$.

It follows from (10) that the reflected solution is

$$\begin{aligned} \psi_R(x,y) = &-\frac{\nu}{\pi i}\int_0^\infty \frac{k^{2\nu-1}(c^{2\nu}-c^{-2\nu})}{(k^{2\nu}-c^{2\nu})\,(k^{2\nu}-c^{-2\nu})}\sin\lambda k y \exp\left(i\lambda k\frac{x}{\gamma}\right) dk \qquad (13) \\ &-\tfrac{1}{2}\sin\lambda c y \exp\{i\lambda c x/\gamma\} - \tfrac{1}{2}\sin\lambda c^{-1} y \exp\{i\lambda c^{-1}(x/\gamma)\}, \end{aligned}$$

where the singular integral is a Cauchy principal value. In the preceding work extensive use has been made of Erdelyi *et al.* (1954), from which the Mellin transform pairs were taken.

The true nature of the solution is not conveyed by (13). The amplitude function has singularities which, as they stand, are not easily interpretable. To overcome this, the singularity in the integral may be subtracted in a suitable manner. With a little manipulation the solution is expressible as

$$\begin{aligned} \psi_R(x,y) = &\text{ ray solution} \\ &-\frac{1}{\pi i}\int_0^\infty\left[\frac{k^{2\nu-1}\nu(c^{2\nu}-c^{-2\nu})}{(k^{2\nu}-c^{2\nu})\,(k^{2\nu}-c^{-2\nu})}-\frac{k(c^2-c^{-2})}{(k^2-c^2)\,(k^2-c^{-2})}\right]\sin\lambda k y\exp\left(i\lambda k\frac{x}{\gamma}\right)dk \\ &+\frac{c-c^{-1}}{\pi}\int_0^\infty \frac{\sin\lambda k y \sin\lambda k(x/\gamma)}{(k+c)\,(k+c^{-1})}\,dk, \qquad (14) \end{aligned}$$

where $\qquad c = \dfrac{\gamma-\alpha}{\gamma+\alpha}, \quad K = \ln c, \quad \nu = \dfrac{\pi}{\pi - iK}.$

The subtraction of the singularity is evident from the first integral in (14). All the integrands are now absolutely integrable. It was shown earlier that the ray solution does not satisfy the radiation condition and hence it should be noted that part of the second integral in (14), namely the part,

$$-\frac{(c-c^{-1})}{2\pi i}\int_0^\infty \frac{\sin \lambda k y \exp\{-i\lambda k(x/\gamma)\}}{(k+c)(k+c^{-1})}\,dk,$$

cancels those modes present in the ray solution which do not satisfy this condition.

It is possible to express the solution in another form by deforming the path of integration in (13). If we introduce the characteristic co-ordinates,

$$\xi = y+\frac{x}{\gamma}, \quad \eta = y-\frac{x}{\gamma},$$

so that $0 \leqslant \xi < \infty$, and $-\infty < \eta < \infty$ in the region of interest, then the total solution may be written as

$$\begin{aligned}\psi_T &= \text{ray solution} + \text{incident wave}\\ &\quad -\frac{\nu}{2\pi}\int_0^\infty \frac{k^{2\nu-1}(c^{3\nu}-c^{-\nu})\,e^{-k\lambda\xi}}{(k^{2\nu}+c^{3\nu})(k^{2\nu}+c^{-\nu})}dk\\ &\quad +\begin{cases}\dfrac{\nu}{2\pi}\displaystyle\int_0^\infty \frac{k^{2\nu-1}(c^{\nu}-c^{-3\nu})\,e^{-k\lambda\eta}}{(k^{2\nu}+c^{\nu})(k^{2\nu}+c^{-3\nu})}dk, & (\eta > 0)\\ \dfrac{\nu}{2\pi}\displaystyle\int_0^\infty \frac{k^{2\nu-1}(c^{3\nu}-c^{-\nu})\,e^{-k\lambda|\eta|}}{(k^{2\nu}+c^{3\nu})(k^{2\nu}+c^{-\nu})}dk, & (\eta < 0).\end{cases}\end{aligned} \tag{15}$$

5. Asymptotic expansions of the solution

In §5 we shall find the asymptotic behaviour of the flow in various regions.

For large values of the characteristic co-ordinates ξ and η, (15) readily yields, using standard expansion techniques,

$$\begin{aligned}\psi_T &= \text{ray solution} + \text{incident wave}\\ &\quad \frac{-\nu(c^{2\nu}-c^{-2\nu})\Gamma(2\nu)}{2\pi}\left[\frac{c^{-\nu}}{(\lambda\xi)^{2\nu}}-\frac{c^{\nu\,\mathrm{sgn}\,\eta}}{(\lambda|\eta|)^{2\nu}}\right]\\ &\quad +O\left(\frac{1}{(\lambda\xi)^{4\nu}}, \frac{1}{(\lambda|\eta|)^{4\nu}}\right), \quad \text{as} \quad \lambda\xi, \lambda|\eta| \to \infty.\end{aligned} \tag{16}$$

Hence the solution obtained here (i.e. the solution satisfying the radiation condition) is asymptotically equivalent to the ray solution at large distances from the origin outside any sector fully containing the characteristic $\eta = y-(x/\gamma) = 0$.

Levey & Mahony (1968) have demonstrated a method of obtaining asymptotic expansions of Fourier integrals over the semi-infinite range valid for small values of the argument. This method involves splitting the integral into two parts, the finite and infinite parts. By adapting this method to the integrals in (15) it is found, for small values of $\lambda\xi$ and $\lambda\eta$, that the contribution from the finite part of the integrals consists of terms like $(\lambda\xi)^n$ and $(\lambda\pi)^n$ $(n = 0, 1, 2, \ldots)$, which exactly cancel the corresponding terms in the Taylor series expansion of the incident wave and the ray solution. The contribution from the infinite part of the integrals then gives rise to the asymptotic behaviour

$$\psi_T \sim \frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\Gamma(2n\nu)}[(\lambda\xi)^{2n\nu}c^{n\nu}-|\lambda\eta|^{2n\nu}c^{-n\nu\,\mathrm{sgn}\,\eta}], \tag{17}$$

as ξ and $\eta \to 0$.

It is evident from (17) that when $\mathrm{Re}\,\nu < \frac{1}{2}$, i.e. $|K| > \pi$, the velocity in the reflected solution becomes infinite as the characteristic is approached. This occurs when the modulus of the slope of the barrier away from the vertical, viz. $|\alpha|$, is greater than $\gamma \tanh \frac{1}{2}\pi\,(\sim 0{\cdot}917\gamma)$.

It can be shown that this singularity is due to a failure of the linear equations employed here to completely describe the flow field, and that to first order it is admissible. Only a brief outline will be given here.

If we retain the Boussinesq approximation and use a perturbation scheme in ϵ, a small parameter governing the supposed infinitesimal incoming wave, so that the dimensionalised stream function is given by

$$\psi = \frac{\sigma}{\lambda^2}(\epsilon\psi^{(1)}+\epsilon^2\psi^{(2)}+\epsilon^3\psi^{(3)}+\ldots), \tag{18}$$

where $\psi^{(1)}$ is the first-order solution we have found, then a detailed analysis of the full non-linearized equations shows that the terms in this expansion are

$$\left.\begin{aligned} \psi^{(1)} &= O(|\lambda\eta|^{2\nu}),\\ \psi^{(2)} &= O(|\lambda\eta|^{2\nu-1}),\\ \psi^{(3)} &= O(|\lambda\eta|^{2\nu-2}).\\ \vdots\quad & \quad\vdots \end{aligned}\right\} \quad \text{as} \quad |\lambda\eta| \to 0.$$

This, with the expansion (18) for ψ, shows that the perturbation scheme is no longer asymptotic for small ϵ when

$$O(\epsilon|\lambda\eta|^{2\nu}) = O(\epsilon^2|\lambda\eta|^{2\nu-1}),$$

or

$$|\lambda\eta| = O(\epsilon).$$

Hence the asymptotic expansion is not uniformly valid near the characteristic $\eta = 0$, and this region should be depleted by an inner layer of width $O(\epsilon/\lambda)$. It must be shown that this layer does not act as a source of energy, momentum or mass for the remaining flow field. This can be done by applying the non-linearized global conservation laws to a region of width $O(\epsilon/\lambda)$ enclosing the entire characteristic $\eta = 0$, and examining the time-averaged flux into this region. It is found that these fluxes are of smaller order in ϵ than those implied by the linear equations. Hence, to the accuracy of the linear equations which are being employed here, the singularity does not act as a source or sink of energy, momentum or mass, and is therefore acceptable.

6. Discussion

In calculating the amplitude function, the inversion path was taken to be the imaginary axis. That this should be the case can be seen by examining the other possibilities. From (12) it is seen that $\bar{A}(s)$ has singularities at the zeros of

$$\sin\left[\frac{s(\pi - iK)}{2}\right],$$

i.e. at
$$s = \frac{2n\pi}{\pi - iK} = 2n\nu, \quad (n = \pm 1, \pm 2, \ldots).$$

These poles, if included, give rise to the eigensolutions in a wedge discussed by Wunsch (1969). They are of the form,

$$\psi_n = \exp(-in\pi\nu)\,\xi^{2n\nu} - \exp(in\pi\nu \operatorname{sgn}\eta)\,|\eta|^{2n\nu} \quad (n = \pm 1, \pm 2, \ldots).$$

These solutions vanish on the boundaries. That none of them can be added to the solution obtained is easily verified by considering their behaviour in various regions. Obviously, the solutions for $n < 0$ cannot be included, for then we would have an infinite discontinuity in the stream function at $\eta = 0$, which is physically unacceptable. The eigensolution for $n = 1$, on the other hand, is just such that it cancels out the most singular terms in the expansion (17) for small $|\eta|$ and ξ. However, if we allow this cancellation, then the stream function becomes infinite in the far field. But the asymptotic expansion is still invalid at distances $O(\epsilon)$ from the singular characteristic, and now, when the conservation laws are considered on a global scale, they are found not to hold. The solution is unacceptable. The singularity is worse for higher values of n, and hence none of the eigensolutions can be included.

It is of interest to note that, apart from constant multiples, the first correction to the ray solution in the far field is the eigensolution corresponding to $n = -1$, and that the expansion near the corner consists of the eigensolutions corresponding to n positive.

The asymptotic expansions found in § 5 do not show the behaviour of the flow for intermediate values of the co-ordinates. To obtain this explicitly in particular cases, expression (15) has been used to evaluate the total velocity numerically at certain positions. Figures 2(*a*) and 2(*b*) give the velocity V on the beach as a function of the scaled vertical height with beach slopes corresponding to

$$\alpha/\gamma = -0{\cdot}5 \text{ and } \alpha/\gamma = -0{\cdot}95,$$

while figure 3 gives, in the case $\alpha/\gamma = -0{\cdot}5$, the value of $(1/\lambda)(\partial\psi_T/\partial\eta)$, which is part of the velocity tangential to the singular characteristic $\eta = 0$. In each case, the velocity predicted by the ray theory is included for comparison, and it is noted that, at distances of approximately one quarter of a wavelength or more from the singular characteristic, the actual solution is close to the ray solution. Referring to figure 3, we see that the velocity tangential to the singular characteristic in the ray solution has a discontinuity, while the solution we have obtained by imposing the radiation condition tends to smooth out this singularity. Figure 2(*b*) has been included to illustrate the manner in which the velocity becomes infinite as the corner is approached when $|\alpha/\gamma| > \tanh\frac{1}{2}\pi$. Physically, this

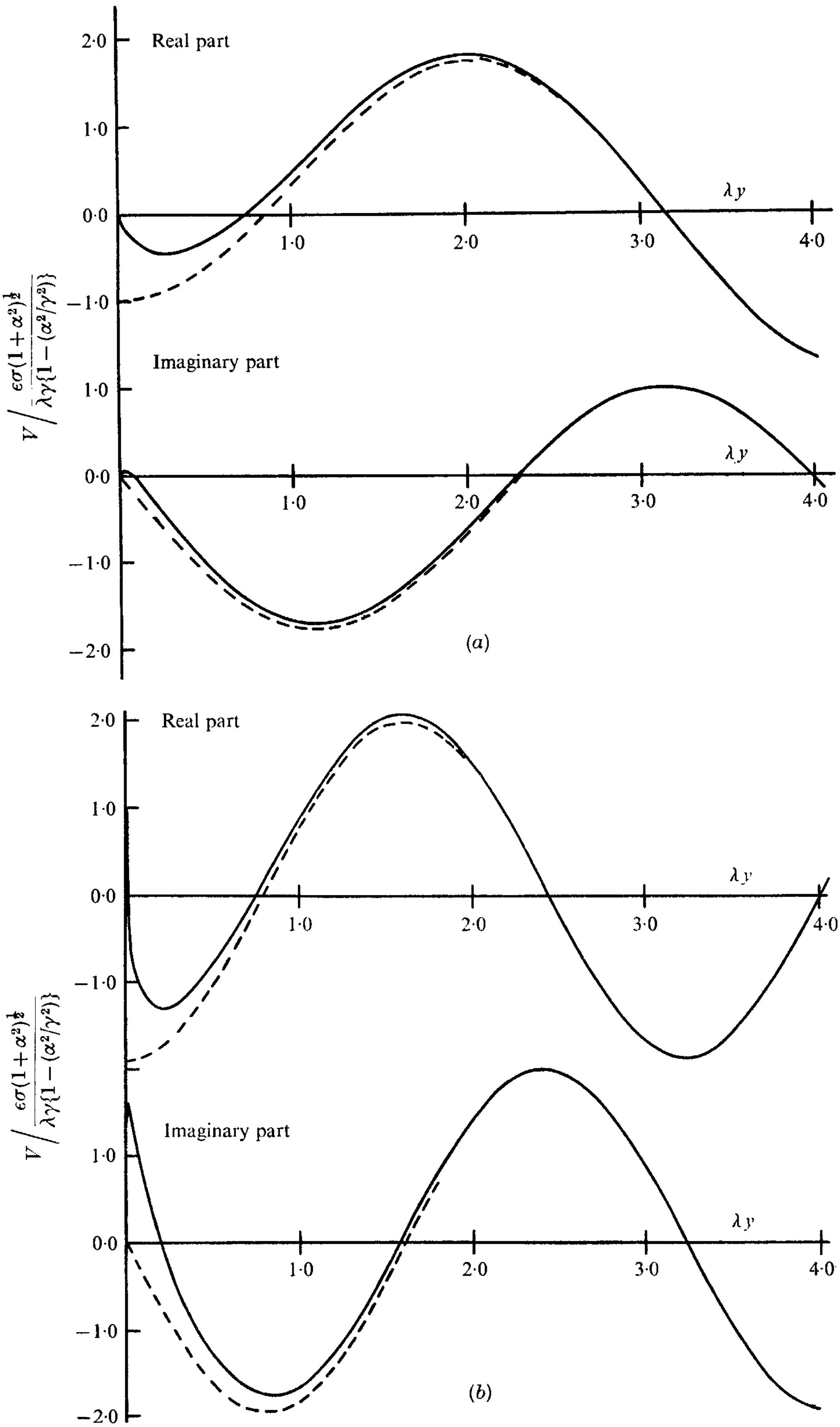

FIGURE 2. The total velocity V on the beach slope as a function of the scaled vertical height λy, (a) in the case $\alpha/\gamma = -0{\cdot}5$, (b) in the case $\alpha/\gamma = -0{\cdot}95$. The dashed line gives the corresponding velocity predicted by the ray theory.

singularity is probably not observable, because, for these slopes, the entire linear theory may have broken down, or, if not, then the above solution is not valid for values of λy less than $O(\epsilon)$, and in this region a smoothing process probably takes place.

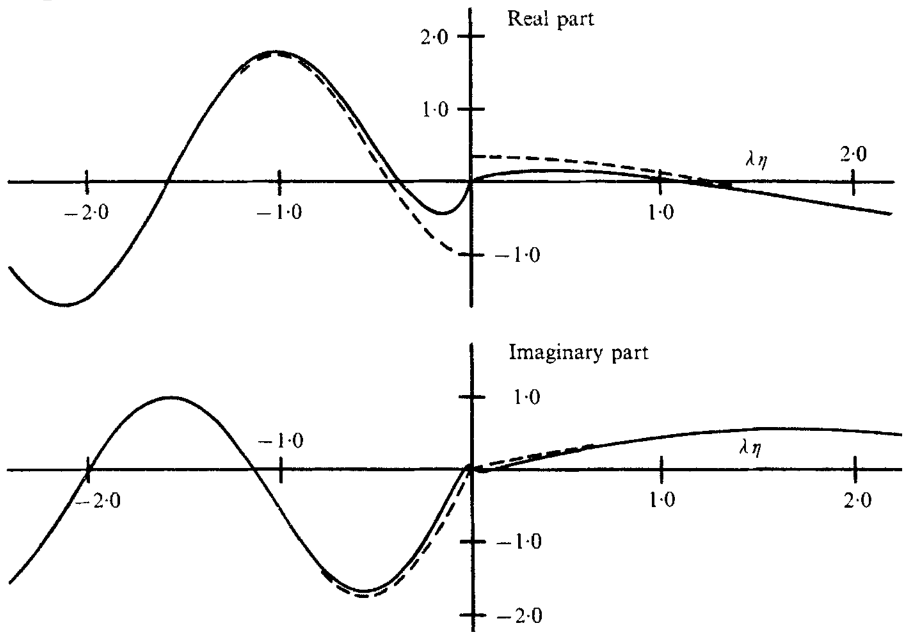

FIGURE 3. The graph, near $\eta = 0$, of $1/\lambda(\partial\psi_T/\partial\eta)$, which is part of the velocity tangential to the singular characteristic $\eta = 0$. $\alpha/\gamma = -0{\cdot}5$. The dashed line gives the corresponding value predicted by the ray theory.

In conclusion, it can be stated that the effect of a corner on a propagating plane wave is that the wave is reflected according to the simple ray theory, except near the corner, where the imposition of the radiation condition smooths out the singularity predicted by this theory.

This work is to be part of a Ph.D. Thesis to be submitted to the University of Western Australia. I gratefully acknowledge assistance and encouragement given to me by my supervisor, Mr D. G. Hurley, and by Professor J. J. Mahony. Part of this work was undertaken while the author was the recipient of a Gledden Fellowship awarded by the University of Western Australia.

Appendix

Here we evaluate

$$A(k) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\sin s[\frac{1}{2}(\pi + iK)]}{\sin s[\frac{1}{2}(\pi - iK)]}\left(\frac{k}{\lambda}\right)^{-s} ds.$$

It is convenient to change variables so that the path of integration is the real axis. By employing the evenness and oddness of the integrand, $A(k)$ reduces to

$$A(k) = -\frac{1}{\pi}\int_0^{\infty} \frac{\sinh s\{\frac{1}{2}(\pi + iK)\}\cos\{s\ln(k/\lambda)\}}{\sinh s\{\frac{1}{2}(\pi - iK)\}} ds.$$

The integrand is not integrable at the upper limit and this singularity must be subtracted to give

$$A(k) = -\frac{1}{\pi}\int_0^\infty \left[\frac{\sinh s\{\frac{1}{2}(\pi+iK)\}}{\sinh s\{\frac{1}{2}(\pi-iK)\}} - e^{iKs}\right]\cos(s\ln k/\lambda)\,ds$$

$$-\frac{1}{\pi}\int_0^\infty e^{iKs}\cos\left(s\ln\frac{k}{\lambda}\right)ds$$

$$= \frac{1}{\pi i}\int_0^\infty \frac{\sin s\{K+\ln(k/\lambda)\}+\sin s\{K-\ln(k/\lambda)\}}{\exp\{s(\pi-iK)\}-1}\,ds$$

$$-\frac{1}{\pi}\int_0^\infty e^{iKs}\cos\{s\ln(k/\lambda)\}\,ds.$$

The first integral in this expression is tabulated in standard tables of Fourier sine transforms, while the second integral requires the generalized oı Cesàro summed integrals:

$$\int_0^\infty \cos kx\,dk = \pi\delta(x), \quad \int_0^\infty \sin kx\,dk = \frac{1}{x}.$$

The resulting $A(k)$ is

$$A(k) = \frac{v}{2\pi i}[\coth\nu\{K+\ln(k/\lambda)\}+\coth\nu\{K-\ln(k/\lambda)\}]$$

$$-\tfrac{1}{2}[\delta\{K+\ln(k/\lambda)\}+\delta\{K-\ln(k/\lambda)\}]$$

where
$$\nu = \frac{\pi}{\pi-iK}.$$

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